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Approximating natural connectivity of scale-free networks based on largest eigenvalue

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Abstract – It has been recently proposed that natural connectivity can be used to efficiently characterize the robustness of complex networks. The natural connectivity has an intuitive physical meaning and a simple mathematical formulation, which corresponds to an average eigenvalue calculated from the graph spectrum. However, as a network model close to the real-world system that widely exists, the scale-free network is found difficult to obtain its spectrum analytically. In this article, we investigate the approximation of natural connectivity based on the largest eigenvalue in both random and correlated scale-free networks. It is demonstrated that the natural connectivity of scale-free networks can be dominated by the largest eigenvalue, which can be expressed asymptotically and analytically to approximate natural connectivity with small errors. Then we show that the natural connectivity of random scale-free networks increases linearly with the average degree given the scaling exponent and decreases monotonically with the scaling exponent given the average degree. Moreover, it is found that, given the degree distribution, the more assortative a scale-free network is, the more robust it is. Experiments in real networks validate our methods and results.

Introduction. – Many systems in the real world can be described as complex networks. The investigation of complex networks has recently become one of the most popular topics in interdisciplinary area [1–3]. In particular, robustness, *i.e.*, the ability of a network to maintain its connectivity when a fraction of its vertices is damaged, is a key aspect of the performance of such networks and has received increasing attention [4–8].

Simple and effective measures are essential for the study of network robustness. A variety of measures, based on different heuristics, have been introduced to quantify the robustness of networks. For example, the vertex (edge) connectivity of a graph, as an important and probably the earliest measure of network robustness [9]. However, it may partially reflect the ability of graphs to retain connectedness after vertexes (or edges) deletion. Besides, another remarkable measure used to characterize the robustness of a network is the second smallest (first nonzero) eigenvalue of the Laplacian matrix, also known as the algebraic connectivity. Fiedler [10] showed that the magnitude of the algebraic connectivity reflects how well the overall graph is connected, *i.e.*, the larger the algebraic connectivity is, the more difficult it is to cut a graph into independent components. However, the algebraic connectivity is equal to zero for all disconnected networks making it too coarse as a measure of robustness. An alternative formulation of robustness within the context of complex networks emerged from random graph theory and was stimulated by the work of Albert *et al.* [4]. They proposed a statistical method to characterize the robustness of complex networks instead of a strict external property. In a recent work, Schneider [11] proposed a new measure R, which considers not only the critical removal fraction when the network is collapsed, but also the size of the largest connected cluster during the malicious attack. Although the critical removal fraction can be obtained analytically for some special networks [12–16], generally this measure can only be calculated through costly simulations.

In our previous works [17,18], the concept of natural connectivity was proposed as a spectral measure of structural robustness in complex networks. Based on the Estrada index of a graph [19,20], the natural connectivity

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is applied here to characterize the robustness of the network which has been used in several contexts of graph theory. Compared with those measures discussed above, the natural connectivity has an intuitive physical meaning and a simple mathematical formulation. Hence, the natural connectivity we provided sets up a bridge between the graph spectrum and the robustness of complex networks and has received growing attention in applications [21–23].

We have studied the natural connectivity of regular ring lattices and random graphs in refs. [24,25], respectively. Here, we focus on the natural connectivity of scale-free networks with power-law degree distributions $p(k) \sim k^{-\gamma}$, which exists in the real world extensively. In the past decade, the ubiquity of scale-free networks has received growing attention. Examples of scale-free networks include the Internet, social networks, or biological networks. Although there are some research results about the spectrum of scale-free networks, so far we still cannot obtain all the exact eigenvalues of scale-free networks analytically. In this article, we will address the approximation problem of natural connectivity in scale-free networks.

The definition of natural connectivity. – A complex network can be described as a simple undirected graph $G = (V, E)$, where V is the set of vertices, and $E \subseteq V \times V$ is the set of edges. Let $N = |V|$ and $W = |E|$ be the number of vertices and the number of edges, respectively. Let d_i be the degree of node v_i, m the minimum degree and M the maximum degree of G. Let $p(k)\,(m \le k \le M)$ be the degree distribution. If the degree distribution follows a power law, *i.e.*, $p(k) \sim k^{-\gamma}$, G is called a scale-free network with scaling exponent γ. We focus on scale-free networks in this study due to their ubiquity. The connectivity of the graph G can be represented by the adjacency matrix $A(G) = (a_{ij})_{N\times N}$, where $a_{ij} = a_{ji} = 1$ if vertex v_i and v_j are adjacent, otherwise $a_{ij} = a_{ji} = 0$. It follows immediately that $A(G)$ is a real symmetric matrix with real eigenvalues $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$, which are usually called the eigenvalues of the graph G itself. The set $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is called the spectrum of G.

The natural connectivity of G is defined as follows [17]

$$\bar{\lambda} = \ln\left(\frac{S}{N}\right) = \ln\left(\frac{1}{N}\sum_{i=1}^{N} e^{\lambda_i}\right), \qquad (1)$$

which corresponds to an average eigenvalue of the graph adjacency matrix. It characterizes the redundancy of alternative paths by quantifying the weighted number of closed walks of all lengths.

Approximation of natural connectivity in random scale-free networks. – There are some research results about the spectrum of scale-free networks, however, we still can not obtain all the exact eigenvalues of scale-free networks analytically. Here, we first consider the approximation expression of natural connectivity in random scale-free networks, in which nodes are connected randomly given a power-law degree distribution.

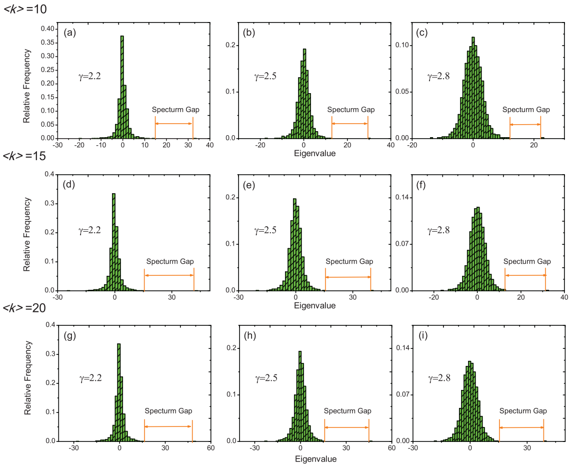

Fig. 1: (Colour online) The histograms of eigenvalues of random scale-free networks with various γ and $\langle k\rangle$, where $N = 1000$. The spectral gap between the two largest eigenvalues is also shown. We also do the experiments in networks with different sizes and observe the same phenomenon.

Using the extended random graph model with described in ref. [26], we generate random scale-free networks with a given expected degree sequence $w_1 \ge w_2 \ge \ldots \ge w_N$, where $w_i = ci^{-1/(\gamma-1)}$, $\gamma > 2$. Here c can be determined by the minimum expected degree $m = w_N = cN^{-1/(\gamma-1)}$, then we obtain that $c = mN^{1/(\gamma-1)}$. It also follows that the maximum expected degree $M = w_1 = mN^{1/(\gamma-1)}$. It is easy to verify that the degree distribution is $p(k) = (\gamma-1)m^{\gamma-1}k^{-\gamma}(m \le k \le M)$ and the average expected degree is $\langle k\rangle = m(\gamma-1)/(\gamma-2)$ [27].

To explore the graph spectrum of scale-free networks, we show the histograms of eigenvalues of random scale-free networks with various scale exponents in fig. 1. It can be observed that, aside from the largest eigenvalue, all the other eigenvalues are contained in the bulk part of the spectrum. It means that there exists an evident spectral gap between the largest eigenvalue and the second largest eigenvalue [28]. The large spectral gap leads to the fact $e^{\lambda_1} \gg e^{\lambda_2} \ge \ldots e^{\lambda_N}$. Hence, we consider the following approximation of the natural connectivity of random scale-free networks based on the largest eigenvalue:

$$\begin{aligned}\bar{\lambda} &= \left(\sum_{i=1}^{N} e^{\lambda_i}/N\right)\\ &= \ln\left[\left(\sum_{i=2}^{N} e^{\lambda_i} + e^{\lambda_1}\right)\Big/N\right]\\ &\approx \lambda_1 - \ln N \equiv \bar{\lambda}^{(\mathrm{I})}.\end{aligned} \qquad (2)$$

On the largest eigenvalues λ_1 of random scale-free networks, there are very few exact analytical results. In ref. [29], Chung *et al.* proved that if $\tilde{d} > \sqrt{M}\ln N$, it is almost sure that $\lambda_1 = \tilde{d}$; if $\sqrt{M} > \tilde{d}\ln^2 N$, it is almost

$$
\begin{aligned}
\bar{\lambda} &\approx \tilde{d} - \ln N \\
&= \begin{cases}
m\dfrac{\gamma-2}{\gamma-3} - \ln N = \langle k \rangle \dfrac{(\gamma-2)^2}{(\gamma-1)(\gamma-3)} - \ln N, & \gamma > 3, \\
m\dfrac{\ln N}{2} - \ln N = \langle k \rangle \dfrac{\ln N}{4} - \ln N, & \gamma = 3, \\
m\dfrac{\gamma-2}{3-\gamma} N^{\frac{3-\gamma}{\gamma-1}} - \ln N = \langle k \rangle \dfrac{(\gamma-2)^2}{(\gamma-1)(3-\gamma)} N^{\frac{3-\gamma}{\gamma-1}} - \ln N, & 2 < \gamma < 3
\end{cases} \\
&\equiv \bar{\lambda}^{(\mathrm{III})}.
\end{aligned}
\tag{10}
$$

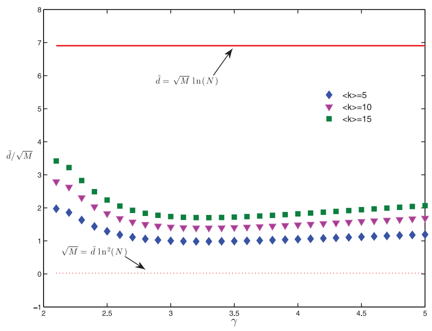

Fig. 2: (Colour online) The ratio between $\tilde{d}$ and $\sqrt{M}$. The solid line represents the boundary condition $\tilde{d} = \sqrt{M}\ln(N)$. The dashed line represents the boundary condition $\sqrt{M} = \tilde{d}\ln^2(N)$. The network size $N = 1000$. We also do the experiments in networks with different sizes and observe the same phenomenon.

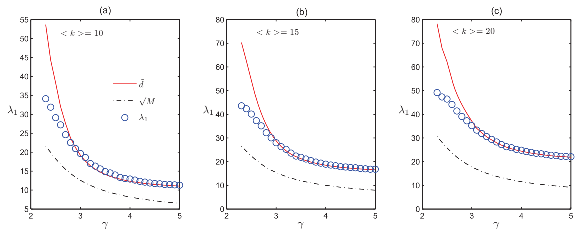

Fig. 3: (Colour online) Approximate estimations of the largest eigenvalues λ_1 of random scale-free networks using the second-order average expected degree $\tilde{d}$ and the square root of maximum expected degree $\sqrt{M}$, respectively. The network size $N = 1000$. We also do the experiments in networks with different sizes and observe the same phenomenon.

sure that $\lambda_1 = \sqrt{M}$, where $\tilde{d} = \langle k^2 \rangle / \langle k \rangle$ represents the second-order average expected degree. We remark that both the conditions $\tilde{d} > \sqrt{M} \ln N$ and $\sqrt{M} > \tilde{d}\ln^2 N$ are actually very harsh, because in most cases the two values of $\tilde{d}$ and $\sqrt{M}$ are not far apart as shown in fig. 2. Thus the two conditions are very difficult to be satisfied.

In fig. 3, we show $\tilde{d}$ and $\sqrt{M}$ along with the largest eigenvalues λ_1 as a function of the scaling exponent γ with different average expected degree $\langle k \rangle$. We find that the largest eigenvalues λ_1 can be well estimated by $\tilde{d}$ rather than $\sqrt{M}$, especially if $\gamma \geq 3$. The observation leads us to consider using $\tilde{d}$ to approximate the largest eigenvalues λ_1. Then we obtain that

$$\bar{\lambda} \approx \lambda_1 - \ln N \approx \tilde{d} - \ln N \equiv \bar{\lambda}^{(\mathrm{II})}. \tag{3}$$

Now we derive $\tilde{d}$ analytically.
When $\gamma > 3$, we can obtain

$$\tilde{d} = \frac{\langle k^2 \rangle}{\langle k \rangle} = m\frac{\gamma-2}{\gamma-3}\frac{N^{\frac{3-\gamma}{\gamma-1}}-1}{N^{\frac{2-\gamma}{\gamma-1}}-1}. \tag{4}$$

Note that $N^{\frac{3-\gamma}{\gamma-1}} \to 0$ and $N^{\frac{2-\gamma}{\gamma-1}} \to 0$ when $N \to \infty$. Thus, we can simplify eq. (4) as

$$\tilde{d} = \frac{\langle k^2 \rangle}{\langle k \rangle} \approx m\frac{\gamma-2}{\gamma-3} = \langle k \rangle \frac{(\gamma-2)^2}{(\gamma-1)(\gamma-3)}. \tag{5}$$

When $\gamma = 3$, we can obtain

$$\tilde{d} = \frac{\langle k^2 \rangle}{\langle k \rangle} = m\frac{\ln N}{2(1-N^{-1/2})}. \tag{6}$$

Note that $N^{-1/2} \to 0$ when $N \to \infty$, thus we can simplify eq. (6) as

$$\tilde{d} = m\frac{\ln N}{2(1-N^{-1/2})} \approx m\frac{\ln N}{2} = \langle k \rangle \frac{\ln N}{4}. \tag{7}$$

When $2 < \gamma < 3$, we can obtain

$$\tilde{d} = \frac{\langle k^2 \rangle}{\langle k \rangle} = m\frac{\gamma-2}{\gamma-3}\frac{N^{\frac{3-\gamma}{\gamma-1}}-1}{N^{\frac{2-\gamma}{\gamma-1}}-1}. \tag{8}$$

Note that $N^{\frac{2-\gamma}{\gamma-1}} \to 0$ when $N \to \infty$, thus we can simplify eq. (8) as

$$\tilde{d} \approx m\frac{\gamma-2}{3-\gamma}N^{\frac{3-\gamma}{\gamma-1}} = \langle k \rangle \frac{(\gamma-2)^2}{(\gamma-1)(3-\gamma)}N^{\frac{3-\gamma}{\gamma-1}}. \tag{9}$$

Consequently, we obtain the asymptotic analytical expression of the natural connectivity of scale-free networks as follows:

see eq. (10) *above*

It is shown in fig. 4 that the exact natural connectivity $\bar{\lambda}$ of random scale-free networks along with our approximations discussed above. We find that $\bar{\lambda}$ can be estimated by $\bar{\lambda}^{(\mathrm{I})}$ very well for various scaling exponents. In other

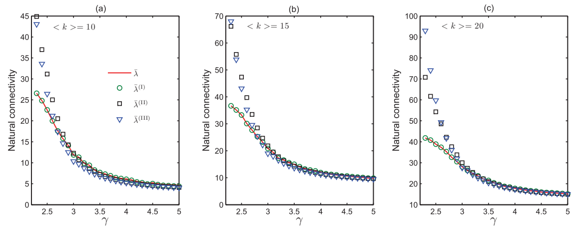

Fig. 4: (Colour online) Natural connectivity of random scale-free networks estimated by $\bar{\lambda}^{(\mathrm{I})}$ in eq. (2), $\bar{\lambda}^{(\mathrm{II})}$ in eq. (3), and $\bar{\lambda}^{(\mathrm{III})}$ in eq. (10) as a function of various exponents γ with different $\langle k\rangle$. The lines represent the exact natural connectivity in eq. (1) and the symbols represent our approximations. The network size $N = 1000$. We also do the experiments in networks with different sizes and observe the same phenomenon.

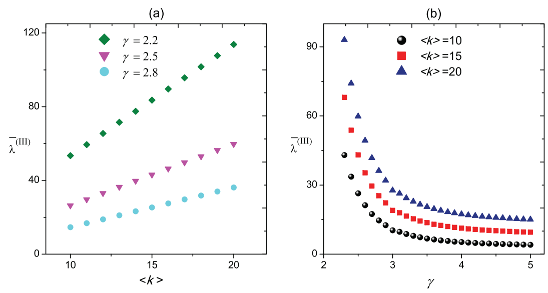

Fig. 5: (Colour online) Natural connectivity of random scale-free networks as a function of average expected degree (a) and scaling exponent (b) based on $\bar{\lambda}^{(\mathrm{III})}$, where $N = 1000$.

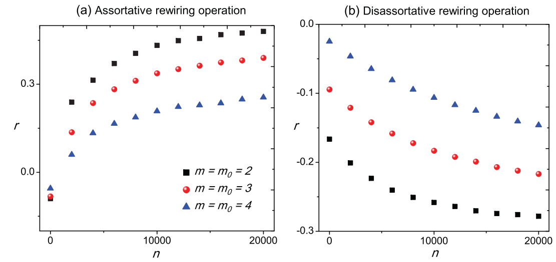

Fig. 6: (Colour online) The degree correlation coefficient r in correlated scale-free networks generated by assortative rewirings (a) and disassortative rewirings (b). The original random scale-free networks are generated using the BA model with various parameters m_0 and m. The network size $N = 1000$.

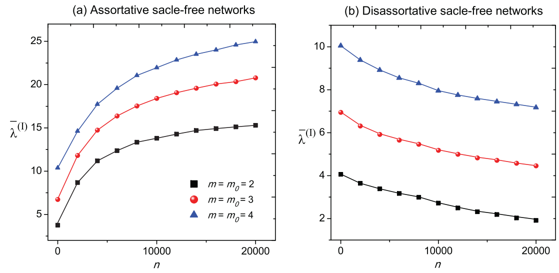

Fig. 7: (Colour online) The natural connectivity as a function of rewiring operation steps in assortative scale-free networks (a) and disassortative scale-free networks (b). The original network is the same network as in fig. 5. The lines represent the exact natural connectivity in eq. (1) and the symbols represent our approximation $\bar{\lambda}^{(\mathrm{I})}$.

words, the natural connectivity can be dominated by the largest eigenvalue in these cases. Besides, when $\gamma \geq 3$, it can be estimated by $\bar{\lambda}^{(\mathrm{II})}$ and $\bar{\lambda}^{(\mathrm{III})}$ with small errors.

Moreover, the mathematical expressions in eq. (10) enable us to easily explore the dependence of the robustness of random scale-free networks on the average expected degree $\langle k\rangle$ and scaling exponent γ. In fig. 5, we show the natural connectivity estimated by $\bar{\lambda}^{(\mathrm{III})}$ as a function of the average expected degree $\langle k\rangle$ and scaling exponent γ, respectively. We find that, given scaling exponent γ, the robustness of random scale-free network is found to increase linearly with average expected degree $\langle k\rangle$; given average expected degree $\langle k\rangle$, the robustness of random scale-free network decreases with scaling exponent γ.

Approximation of natural connectivity in correlated scale-free networks. – We have discussed the approximation of natural connectivity in random scale-free network. However, many networks in real-world show "assortative mixing" on their degree, *i.e.*, high-degree vertices associate preferentially with other high-degree vertices; or "disassortative mixing", *i.e.*, high vertices prefer to attach to low-degree ones. Both situations are defined as degree-degree correlation [30]. It has been shown that the degree correlation can have a substantial effect on the behaviors of networks.

We utilize an edge rewiring operation to change the network degree correlation while keeping its degree sequence and global connectivity constant. We first generate random scale-free networks using the BA model [31], and then generate correlated scale-free networks utilizing the assortative edge rewiring and disassortative edge rewiring while keeping its degree distribution.

In fig. 6, the degree correlation coefficient r [30] is shown as a function of the rewiring operation steps. In fig. 6(a), we observe that the degree correlation coefficient r increases with the assortative rewiring operation steps n, which means that the network tends to be more assortative. While in fig. 6(b), it is shown that the degree correlation coefficient r decreases with the disassortative rewiring operation steps n, which suggests that the network tends to be more disassortative. We show in fig. 7 the exact natural connectivity $\bar{\lambda}$ along with the approximation $\bar{\lambda}^{(\mathrm{I})}$ in eq. (2). We find that our approximations $\bar{\lambda}^{(\mathrm{I})}$ agree well with the exact values $\bar{\lambda}$. Moreover, fig. 7 shows that the natural connectivity of scale-free network increases with the assortative rewiring operation steps n and decreases with disassortative rewiring operation steps n. The results suggest that, given the degree distribution, the more assortative the scale-free network is, the more robust it is.

Table 1: Basic statistics of real networks. V and E are the number of nodes and links. $\langle k \rangle$ is the average degree. C is the clustering coefficient. $\langle l \rangle$ is the average shortest distance. r is the assortativity.

Networks	V	E	$\langle k \rangle$	C	l	r
Email	1133	5451	9.6	0.166	3.65	0.078
DBLP	12591	49743	7.9	0.062	4.42	−0.045
USairports	1574	28236	21.87	0.384	3.14	−0.113
Proteins	2239	6452	5.76	0.007	3.98	−0.33
Advogato	6541	51127	12.62	0.092	3.29	−0.095
Routeviews	6474	13895	4.29	0.009	3.67	−0.182
Linux	30837	230213	7.47	0.002	3.3	−0.174
Lexicon	1773	9131	10.3	0.163	3.38	−0.048
PB	1222	16714	27.36	0.36	2.74	−0.221

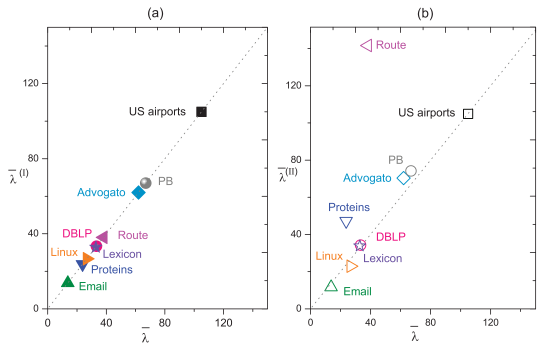

Fig. 8: (Colour online) The natural connectivity of nine real-world networks based on $\bar{\lambda}^{(\mathrm{I})}$(a) and $\bar{\lambda}^{(\mathrm{II})}$(b). The horizontal axis is the exact natural connectivity $\bar{\lambda}$ and the vertical axis is its approximation. Different symbol represents different network.

Experiments in real-world networks. – Many systems taking the form of networks in the real-world, are more complicated than the synthetic networks. To validate our methods and results, we investigate the approximation of natural connectivity in various real-world scale-free networks: an email network (Email), the citation network (DBLP) extracted from a database of scientific publications such as papers and books, the network of flights between US airports in 2010 (USairports), the interactions between proteins in Humans (Proteins), an online community platform (Advogato), the network of autonomous systems of the Internet (Routeviews), the network of Linux source code files (Linux), the lexical network containing nouns including places and names of the King James Bible and information about their occurrences (Lexicon) and the U.S. political blogs (PB) network. All the data can be downloaded from `http://konect.uni-koblenz.de/`. Basic statistics of these networks are shown in table 1.

We show in fig. 8 the exact natural connectivity and the approximations for these real-world networks. In fig. 8(a), it is observed that all the symbols are arranged along the diagonal line. It means that the exact natural connectivity $\bar{\lambda}$ can be estimated by $\bar{\lambda}^{(\mathrm{I})}$ very well in these real-world networks. In fig. 8(b), we find that the approximation $\bar{\lambda}^{(\mathrm{II})}$ based on the random scale-free network model is not as good as $\bar{\lambda}^{(\mathrm{I})}$ and some symbols deviate significantly from the diagonal line. The deviation comes from two aspects. On the one hand, it may come from the approximation errors; on the other hand, it can be explained by the fact that the real networks are more complex than random networks and then the random scale-free model may not characterize them properly.

Conclusions. – In this letter, we have investigated the approximation problem of natural connectivity, which is introduced as a spectral measure of network robustness, in scale-free networks. Based on the observation that the spectrum of scale-free networks has a large gap between the largest eigenvalue and the bulk part of the spectrum, it has been demonstrated that the natural connectivity of scale-free network is dominated by the largest eigenvalue in both random and correlated scale-free networks. The largest eigenvalue of a network, as a key factor, plays an important role in describing the topological and dynamical characteristics of networks [32–34]. For example, it is shown that the epidemic threshold for a network is closely related to the largest eigenvalue of its adjacency matrix in epidemic spreading [35]. The link between robustness and largest eigenvalue is of great theoretical and practical significance in network analysis, as it opens possibilities to connect robustness to other network structural or dynamical properties such as efficiency, synchronization, epidemic spreading, and search ability.

We have presented an approximate analytical expressions of the natural connectivity of random scale-free networks. The proposed approximation agrees with the numerical results well when $\gamma \geq 3$. Based on the approximate analytical expression, we have explored the robustness of random scale-free networks and we have found that it increases linearly with the average degree given the scaling exponent and decreases monotonically with the scaling exponent given the average degree. Moreover, we have shown that, given the degree distribution, the more assortative the scale-free network is, the more robust the scale-free network is. Experiments in real networks validate our methods and results. Our results can be of potential significance for network robustness design and optimization.

* * *

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